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Spin-statistics transmutation in relativistic quantum field theories of dyons

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Abstract: We analyse spin and statistics of quantum dyon fields, i.e. fields carrying both electric and magnetic charge, in $3+1$ space-time dimensions. It has been shown long time ago that, at the quantum mechanical level, a composite dyon made out of a magnetic pole of charge g and a particle of electric charge e possesses half-integral spin and fermionic statistics, if the constituents are bosons and the Dirac quantization condition $eg = 2\pi n$ holds, with n *odd.* This phenomenon is called spin-statistics transmutation. We show that the same phenomenon occurs at the quantum field theory level for an *elementary* dyon. This analysis requires the construction of gauge invariant charged dyon fields. Dirac's proposal for such fields, relying on a Coulomb-like photon cloud, leads to quantum correlators exhibiting an unphysical dependence on the Dirac-string. Recently Froehlich and Marchetti proposed a recipe for charged dyon fields, based on a sum over Mandelstam-strings, which overcomes this problem. Using this recipe we derive explicit expressions for the quantum field theory correlators and we provide a proof of the occurrence of spin-statistics transmutation. The proof reduces to a computation of the self-linking numbers of dyon worldlines and Mandelstam strings, projected on a fixed time three-space. Dyon composites are also analysed. The transmutation discussed in this paper bares some analogy with the appearance of anomalous spin and statistics for particles or vortices in Chern-Simons theories in $2+1$ dimensions. However, peculiar features appear in $3+1$ dimensions e.g. in the spin addition rule.

Keywords: Solitons Monopoles and Instantons, Duality in Gauge Field Theories.

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Contents

1. Introduction

In this paper we analyse spin and statistics of quantum dyon fields, i.e. fields carrying both electric and magnetic charge in $3+1$ space-time dimensions, relying upon the construction of their correlation functions sketched in [1].

Previous discussions about this subject were all based either on a quantum mechanical analysis of *composite* dyons, i.e. composites of a magnetic monopole and a charged particle [2, 3], or on a semiclassical treatment of quantum field theories where dyons appear [4], but never, at least to our knowledge, on an analysis of "elementary dyons" in a fully quantized field theory. With "elementary dyon" we mean a point-like particle which carries electric as well as magnetic charge.

Both at the classical and quantum mechanical levels one finds indications that composite dyons may exhibit anomalous spin. Classically one can compute the angular momentum $\vec{J}$ stored in the electromagnetic field generated by a magnetic pole with magnetic charge g located at the origin and an electric charge of strength e at a distance a along the 3-axis. For the electric field we have $\vec{E} = \frac{e}{4\pi}\frac{\vec{x}-\vec{a}}{|\vec{x}-\vec{a}|^3}$ and for the magnetic field we have $\vec{B} = \frac{g}{4\pi}\frac{\vec{x}}{|\vec{x}|^3}$. The only non-vanishing component of the angular momentum is then

$$J_3 = \int d^3x \left[\vec{x} \wedge (\vec{E} \wedge \vec{B})\right]_3 = \frac{eg}{4\pi}\,, \tag{1.1}$$

where the last equality holds for $a \neq 0$. The quantum mechanical requirement that the total angular momentum is quantized with spectrum contained in $\frac{\mathbb{Z}}{2}(\hbar = 1)$ reproduces the Dirac quantization condition for the coexistence of electrically and magnetically charged particles:

$$eg \in 2\pi\mathbb{Z}\,. \tag{1.2}$$

In particular if $eg/2\pi$ is odd, the classical calculation suggests that the composite dyon carries half-integral spin.

These ideas can be made mathematically precise [5] and it has been proved that the Hilbert space of states of the composite dyon carries a projective representation of the rotation group, provided Dirac's quantization condition (1.2) holds, and under this condition the wave function acquires a phase factor $e^{ieg/2}$ under a 2π-rotation.

Subsequently it has been shown that for such dyons the usual spin-statistics connection holds, so that if the constituent particles are bosons and $eg = 2\pi$ the composite behaves as a fermion with half-integral spin [3, 6]. We call this phenomenon "spin-statistics transmutation" borrowing a terminology frequently used in $(2+1)$-dimensional systems [7].

Let us turn to "elementary" point-like dyons. If we set $a = 0$ in the calculation of the classical angular momentum, we find that the integral in (1.1) vanishes. Hence it is far from clear what happens at the quantum level for elementary dyons. One of the main purposes of this paper is to clarify this issue in the framework of relativistic quantum field theory.

Several problems have to be solved along the way to define a consistent setting. The classical equations of motion of a relativistic point-like dyon (coupled to the electromagnetic field) are Lorentz covariant, reading as

$$\begin{aligned}
\partial^\mu F_{\mu\nu}(x) &= e\int d\tau \frac{dy_\nu(\tau)}{d\tau}\delta^{(4)}(x-y(\tau))\,,\\
\partial^\mu * F_{\mu\nu}(x) &= g\int d\tau \frac{dy_\nu(\tau)}{d\tau}\delta^{(4)}(x-y(\tau))\,,\\
m\frac{d^2y^\mu}{d\tau^2} &= (eF^{\mu\nu} + g * F^{\mu\nu})\frac{dy_\nu}{d\tau}\,,
\end{aligned} \tag{1.3}$$

where $y^\mu(\tau)$ parametrizes the particle trajectory, τ is the proper time, $*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and m is the mass of the dyon. However, the implementation of a Lorentz-invariant action principle, even for classical point-like particles, encounters some difficulty. The main reason for this is that an action principle for Maxwell's equations requires necessarily the introduction of vector potentials. Eventually the action for classical point-like particles turns out to be Lorentz-invariant modulo some integer, if Dirac's condition holds.

The situation becomes even worse at the level of the quantum field theory. In a functional integral approach the fundamental ingredients are a classical *field* theory action and the corresponding classical equations of motion for the dyon fields. Contrary to what happens for the system (1.3) for point-like particles, the equations for dyon fields, e.g. Dirac's equations, involve the vector potential explicitly through a covariant derivative. But this potential is a priori not well defined because the presence of monopole-like configurations introduces unphysical Dirac-strings, which are in a sense "global" gauge artifacts.

As a matter of fact, a consistent classical field theory of dyons does not exist and the possibility of a consistent setting for quantum fields relies upon a realization of field-particle quantum duality [8], which allows, roughly speaking, to write the correlation functions of local (with compact support) gauge-invariant observables in terms of a consistent quantum mechanics of point-like dyons. As shown in [8, 9] these correlation functions are independent of the position of Dirac-strings provided some form of "Dirac quantization condition" is imposed.

It turns out that there are two inequivalent consistent classes of quantum field theories of dyons, characterized by their (S-) duality groups [1]. At the classical level one verifies easily that the equations of motion (1.3) are invariant under an SO(2) group of transformations, parametrized by an angle θ, defined by

$$\begin{aligned} \begin{pmatrix} F^{\mu\nu} \\ *F^{\mu\nu} \end{pmatrix} &\Longrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ *F^{\mu\nu} \end{pmatrix}, \\ \begin{pmatrix} e \\ g \end{pmatrix} &\Longrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix}. \end{aligned} \tag{1.4}$$

Let us consider a quantum field theory of dyons of N species with charges $\{e_r, g_r\}$, $r = 1, \ldots, N$. One can show that there is a consistent quantum field theory preserving only the $\mathbb{Z}_4$ subgroup of SO(2) generated by the rotation of an angle $\theta = \pi/2$ (common to all species), provided the Dirac quantization condition

$$e_r g_s \in 2\pi\mathbb{Z} \tag{1.5}$$

holds.

There is also a consistent quantum field theory preserving the full SO(2) duality group, provided the "Schwinger-Zwanziger" quantization condition

$$\frac{1}{2}(e_r g_s - e_s g_r) \in 2\pi\mathbb{Z} \tag{1.6}$$

holds.

Actually, from the correlation functions of local fields mentioned above one can only reconstruct the neutral observables and the vacuum sector of the quantum field theory, i.e. the physical states which are electrically and magnetically neutral; for the construction of charged field operators and of charged sectors one needs something more.

Owing to a theorem discussed by Strocchi [10], gauge-invariant charged fields must be necessarily non local. A natural prescription for the construction of electrically charged fields was given by Dirac [11], and it corresponds essentially to a gauge-invariant dressing of the non gauge-invariant local charged field (e.g. a scalar $\hat{\phi}$ of charge e in scalar QED), by multiplying it with an exponential of the photon field $\hat{A}$, weighted by a classical Coulomb field $\vec{E}$, generated by a unit charge located at $\vec{x}$:

$$\hat{\phi}(\vec{x}) \Longrightarrow \hat{\phi}(\vec{x})\, e^{ie\int \hat{A}(\vec{y})\cdot\vec{E}(\vec{x}-\vec{y})d^3y}\,. \tag{1.7}$$

However, this procedure becomes inconsistent when electric and magnetic dynamical charges coexist. Indeed, one can show that the correlation functions of the fields dressed in this way depend on the position of the Dirac-strings, even if some quantization condition for the charges is imposed. A proposal for a modification of Dirac's recipe which leads to Dirac-string independent correlation functions has been presented in [12], and in [1] we sketched how this proposal can be adapted to the setting of a quantum field theory of dyons.

Using the charged fields constructed accordingly we will perform in the present paper a derivation of spin and statistics of elementary quantum dyons with the following results: (1) for a $\mathbf{Z}_4$-dyon with charges e_r and g_r such that $e_r g_r/2\pi$ is *odd* spin-statistics transmutation from a boson to a fermion (and viceversa) takes place while, if $e_r g_r/2\pi$ is *even*, there is no transmutation; (2) for an SO(2)-dyon spin and statistics are always the naive ones.

The derivation of these results relies on an explicit representation of euclidean correlation functions of charged fields, in terms of a sum over currents with support on families of loops (closed particles trajectories); both spin and statistics are then related to the self-linking numbers of the projections of these loops on a three-dimensional space at fixed time.

In a certain sense, the regularization needed for the definition of the self-linking numbers plays the role of the finite distance a between the magnetic and electric charge in the classical calculation of the angular momentum for composite dyons, see (1.1). The results we obtain for $\mathbf{Z}_4$-dyons are, in fact, in agreement with the suggestions obtained from those calculations.

For concreteness we will assume throughout this paper that our dyons carry "intrinsic" bosonic statistics, i.e. that they are described by complex scalar fields (with an appropriate soft photon cloud). This means that we are analysing the transmutation from bosons to fermions. The opposite case, where the dyons are

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described by complex spinor fields, can be analysed with the same techniques [8, 13] and leads to the same results.

Finally, let us remark that in this paper we use the formalism of differential forms and that we adopt a functional integral approach to quantum field theory at a formal level, i.e. ignoring ultraviolet divergences. Some U.V. regulator is implicitly understood (e.g. the lattice) and we do not discuss its removal; however, we expect our final results to be stable under renormalization because spin and statistics are large-scale properties, but see also the comments at the end of section 5.

2. Relativistic quantum field theory of dyons

2.1 Classical point-like particles

As discussed in the introduction there are two (in general) inequivalent classes of relativistic quantum field theories (QFT) of dyons classified by their duality groups, $\mathbb{Z}_4$ or SO(2).

Both classes admit different, but equivalent, formulations: à la Schwinger [14] with one gauge potential (with a single string or with two half-weighted antisymmetric strings, respectively), à la Zwanziger [15] (with an $i\varepsilon$-prescription to invert $n_\mu\partial^\mu$ in the gauge propagator or with the principal value prescription, resp.), à la PST [16] (with or without an additional interaction term "$\frac{1}{2}C_1 \wedge C_2''$ between Dirac strings, resp.) or its dual formulation [17].

Although the PST formulation has some advantages in that it is manifestly Lorentz invariant, in this paper we adopt the formulation à la Schwinger for the $\mathbf{Z}_4$-dyons, which are our main concern since they exhibit spin-statistic transmutation; on the other hand, our remarks on the SO(2)-dyons are based on a formulation à la Zwanziger. These formulations are somewhat simpler at the QFT level and this motivates our choice.

We start by introducing some basic notions in the theory of (de-Rham) currents [18] which will be of key importance in the following. A p-current in $\mathbb{R}^4$ is a linear functional on the space of smooth (4-p)-forms with compact support, which is continuous in the sense of distributions; i.e. p-currents are "p-forms" with distribution-valued components.

In the space of currents one can define a map, here denoted by PD, extending Poincaré duality, which associates to every p-dimensional surface Σ_p a (4-p) current Φ_{Σ_p} according to

$$\begin{aligned} \mathrm{PD}:\ \Sigma_p &\longrightarrow \Phi_{\Sigma_p} \equiv \mathrm{PD}(\Sigma_p)\,, \\ \int_{\Sigma_p} \alpha_p \ &= \ \int_{\mathbb{R}^4} \alpha_p \wedge \Phi_{\Sigma_p}\,, \end{aligned} \tag{2.1}$$

for any smooth p-form α_p of compact support. In particular, the image by PD of a closed surface is a closed current i.e. if $\partial\Sigma_p = \emptyset$ then $d\Phi_{\Sigma_p} = 0$, where ∂ denotes the boundary operator. Via regularization one can define also the integral of a current Φ_{Σ_p} over a generic $4-p$ dimensional surface S_{4-p}, and the result is an integer if the integral is well defined and it counts the number of intersection with sign, m, of Σ_p and S_{4-p}. We can then formally write

$$\int_{\mathbb{R}^4} \Phi_{\Sigma_p} \wedge \Phi_{S_{4-p}} = m\,. \tag{2.2}$$

Linear combinations of such p-currents with integer coefficients are called integer currents; they are PD-dual to integer (4-p)-chains [18].

After this digression we turn to dyons. Let $\{\gamma_r \equiv y_r^\nu(\tau_r)\}$ denote a set of boundaryless worldlines of dyons with charges $\{e_r, g_r\}$, parametrized by proper times τ_r,. To such trajectories one can associate 3-currents

$$J_{(r)}(x) = \frac{1}{3!} dx^\mu \wedge dx^\nu \wedge dx^\rho \varepsilon_{\mu\nu\rho\sigma} \int d\tau_r \frac{dy_r^\sigma}{d\tau_r}(\tau_r)\delta^4\left(x - y_r(\tau_r)\right), \tag{2.3}$$

which are PD-dual to the support of the worldlines in $\mathbb{R}^4$, i.e. $J_{(r)} = \mathrm{PD}(\gamma_r)$. The total electric and magnetic currents generated by the dyons are given by[1]

$$J_1 = \sum_r e_r J_{(r)}, \qquad J_2 = -\sum_r g_r J_{(r)}\,. \tag{2.4}$$

The electric and magnetic currents for each species are individually conserved and this is expressed through the equations

$$dJ_{(r)} = 0\,. \tag{2.5}$$

Since $\mathbb{R}^4$ is contractible, one can apply the Poincaré lemma (for currents [18]) and (2.5) implies the existence of a 2-current $C_{(r)}$ such that $J_{(r)} = dC_{(r)}$. Actually one can choose $C_{(r)}$ in a more specific form: from (2.5) and $J_{(r)} = \mathrm{PD}(\gamma_r)$ one derives that γ_r is the boundary of a 2-surface Σ_r and we can choose

$$C_{(r)} = \mathrm{PD}(\Sigma_r)\,. \tag{2.6}$$

We then set

$$C_1 = \sum_r e_r C_{(r)}\,, \qquad C_2 = -\sum_r g_r C_{(r)}\,, \tag{2.7}$$

and we obtain

$$J_1 = dC_1\,, \qquad J_2 = dC_2\,. \tag{2.8}$$

If γ_r describes the world line of a magnetic pole, one can think of Σ_r as the support of the world-surface swept out by its Dirac string.

[1] The minus sign in the definition of J_2 is due to our conventions for differential forms. The differential d acts from the right, $d(\phi_p \wedge \phi_q) = \phi_p \wedge d\phi_q + (-)^q d\phi_p \wedge \phi_q$, and the components of a form are defined by $\phi_p = \frac{1}{p!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \phi_{\mu_p \cdots \mu_1}$.

Within this set up the classical action proposed by Schwinger [14] to derive the Maxwell-Dirac equations (1.3) is given by

$$S_S(A, J_1, C_2) = \int \frac{1}{2}(dA + C_2) \wedge *(dA + C_2) + A \wedge J_1 \,, \tag{2.9}$$

where A is a 1-form describing the electromagnetic gauge potential. Independence of the choice of the "Dirac-string" $C_{(r)}$, satisfying $dC_{(r)} = J_{(r)}$, is a consistency condition for the theory described by the action (2.9).

At the quantum level the effective action S_{eff} is defined by

$$e^{iS_{\text{eff}}(J_1,C_2)} = \int \mathcal{D}A e^{iS_S(A,J_1,C_2)} \,. \tag{2.10}$$

A simple calculation proves that

$$S_{\text{eff}}(J_1, C_2) = \int -\frac{1}{2}\left(J_1 \wedge *\Box^{-1} J_1 + J_2 \wedge *\Box^{-1} J_2\right) + J_1 \wedge \delta\Box^{-1} C_2 \,, \tag{2.11}$$

where $\Box = \delta d + d\delta$ is the d'alambertian and $\delta = *d*$ is the codifferential. Consistency requires that $e^{iS_{\text{eff}}(J_1,C_2')} = e^{iS_{\text{eff}}(J_1,C_2)}$, where C_2' is a new Dirac-string, i.e. the exponentiated effective action has to be independent of the choice of Dirac-string. The new Dirac-strings are represented by surfaces Σ_r', whose boundaries are again γ_r. We can always write

$$C_{(r)}' = C_{(r)} + dH_{(r)} \,, \tag{2.12}$$

where $H_{(r)} = \mathrm{PD}(V_r)$, V_r being a 3-volume bounded by $\Sigma_r' - \Sigma_r$, and $C_{(r)}' = \mathrm{PD}(\Sigma_r')$. This leads to

$$\begin{aligned} C_2' &= C_2 + dH_2 \,, \\ H_2 &= -\sum_r g_r H_{(r)} \,. \end{aligned} \tag{2.13}$$

From (2.11), using $dJ_1 = 0$, one obtains

$$\begin{aligned} S_{\text{eff}}(J_1, C_2') - S_{\text{eff}}(J_1, C_2) &= \int J_1 \wedge \delta\Box^{-1} dH_2 = \int J_1 \wedge H_2 \\ &= -\sum_{r,s} e_r g_s \int J_{(r)} \wedge H_{(s)} \,. \end{aligned} \tag{2.14}$$

Since $\int J_{(r)} \wedge H_{(s)}$ is integer, consistency is achieved if

$$e_r g_s = 2\pi n_{rs} \,, \tag{2.15}$$

i.e. if Dirac's quantization condition holds. This turns out to be also the condition to implement the $\mathbf{Z}_4$-duality group at the quantum level. In fact the generator of $\mathbb{Z}_4$ maps $S_{\text{eff}}(J_1, C_2)$ to $S_{\text{eff}}(J_1, C_2) + \int C_1 \wedge C_2$, see [1]. The last term is irrelevant if

$$\int C_1 \wedge C_2 = \sum_{r,s} e_r g_s \int C_{(r)} \wedge C_{(s)} \in 2\pi\mathbb{Z} \,. \tag{2.16}$$

Since the integrals are integer this condition is fulfilled thanks to (2.15).

2.2 Quantum field theory

Next we extend these ideas to a quantum field theory setting. Following an intuition due essentially to Zwanziger [8, 15], the key ingredient of the consistency check of a QFT of dyons is a representation of correlation functions of local observables in terms of Feynman path integrals over closed classical currents, coupled to the gauge field. This procedure corresponds to a realization of quantum field-particle duality.

In order to be selfcontained and to set up the notations, let us give an example of this procedure in a simplified model: a complex scalar field ϕ of charge e coupled to the gauge field A, without additional interactions. The action of the system is given by $S(A) + S(A, \phi)$, where

$$S(A, \phi) = -\int d^4x\, \bar{\phi}\left(\Box_{eA} + m^2\right)\phi\,, \tag{2.17}$$

$\bar{\phi}$ denotes the complex conjugate field and $\Box_{eA}$ is the covariant d'alambertian. $S(A)$ denotes the free Maxwell action. Consider the gauge-invariant correlation function

$$\left\langle T\,\phi(x)\, e^{ie\int j_{xy}\wedge A}\, \bar{\phi}(y)\right\rangle(A)\,, \tag{2.18}$$

where $\langle\,\rangle(A)$ denotes the expectation value with respect to the "measure" in the path-integral induced by $S(A, \phi)$, with A treated as an external field. j_{xy} is the PD of a curve joining x to y.

The relevant Feynman-Schwinger representation can be derived formally as follows:

$$\begin{aligned}\left\langle T\,\phi(x)\bar{\phi}(y)\right\rangle(A) &= \left(\Box_{eA} + m^2 + i\varepsilon\right)^{-1}(x, y)\\ &= i\int_0^\infty ds\, e^{is(m^2+i\varepsilon)}(e^{-is\Box_{eA}})(x, y)\,.\end{aligned} \tag{2.19}$$

The operator $\Box_{eA}$ can be regarded as the "hamiltonian" of a particle of mass $1/2$ and charge e in $3+1$ dimensions, minimally coupled to the gauge field A. The last term in (2.19) can then be viewed as the evolution kernel of this particle with initial position x at "time" 0 and final position y at "time" s. It allows, therefore, a Feynman path integral representation:

$$\left(e^{-is\Box_{eA}}\right)(x, y) = \int \mathcal{D}y(\tau)\, e^{i\int_0^s d\tau\left[\frac{1}{4}\dot{y}^2(\tau) + e\dot{y}_\mu(\tau)A^\mu(y(\tau))\right]}\,. \tag{2.20}$$

Associating to the trajectory $\{y(\tau)\}$, which starts from x and ends in y, a current J as in (2.3) one can write for a suitable "measure" $\mathcal{D}\mu(J)$:

$$\left\langle T\,\phi(x)e^{ie\int j_{xy}\wedge A}\bar{\phi}(y)\right\rangle(A) = \int \mathcal{D}\mu(J)\, e^{ie\int(j_{xy}+J)\wedge A}\,. \tag{2.21}$$

Notice, in particular, that $d(j_{xy}+J)=0$. Similarly for the partition function $Z(A)$ of the field ϕ one can write:

$$Z(A) = \int \mathcal{D}\phi\, e^{-i\int \bar{\phi}(\Box_{eA}+m^2+i\varepsilon)\phi} = \det{}^{-1}(\Box_{eA}+m^2+i\varepsilon) \tag{2.22}$$
$$= \exp\left[-\int d^4x \int_0^\infty \frac{ds}{s}\, e^{i(m^2+i\varepsilon)s}(e^{-is\Box_{eA}})(x,x)\right] = \int \mathcal{D}\mu(\mathbf{J}) e^{ie\int A\wedge \mathbf{J}}\,,$$

for a suitable "measure" $\mathcal{D}\mu(\mathbf{J})$ on networks $\mathbf{J}$ of *closed* currents.

Finally for the normalized correlation functions we have

$$\left\langle T\phi(x)e^{ie\int j_{xy}\wedge A}\bar{\phi}(y)\right\rangle = \frac{\int \mathcal{D}A\, e^{iS(A)} \int \mathcal{D}\mu(\mathbf{J})\, \mathcal{D}\mu(J)\, e^{ie\int(j_{xy}+J+\mathbf{J})\wedge A}}{\int \mathcal{D}A\, e^{iS(A)}\int \mathcal{D}\mu(\mathbf{J})\, e^{ie\int \mathbf{J}\wedge A}}\,, \tag{2.23}$$

which is the representation we will need for our purposes.

We present now the construction of a consistent quantum field theory of $\mathbf{Z}_4$-dyons, where the dyon fields are a family of complex scalars $\{\phi_r\}$ with charges $\{e_r, g_r\}$, interacting with a gauge field A. The basic idea is to start from the Schwinger action (2.9) and to promote C_2 to a real field variable, obeying the constraint

$$dC_2 = J_2\left(\{\phi_r\}, A\right), \tag{2.24}$$

where $J_2(\{\phi_r\},A)$ is the Hodge dual of the total magnetic current generated by the fields $\{\phi_r\}$, i.e. of $-i\sum_r g_r\bar{\phi}_r D_r^\mu\phi_r$ + c.c. The covariant derivative appearing here will be specified below. Eventually we will apply field/particle duality to prove the consistency of the theory, provided Dirac's quantization condition (2.15) holds.

The problem related with the constraint (2.24) is that it does not specify completely the field C_2: this constraint determines C_2 only modulo exact forms. To determine this field completely we proposed in [1] to modify the Schwinger action as follows: one introduces a constant vector u^μ satisfying $u^2 \neq 0^2$ and the Lagrange multiplier fields A_1, a real 1-form, and C_1, a real 2-form. Setting $A \equiv A_2$ we define the QFT Schwinger action, which depends also on the constant vector u, as

$$S_S^u\left(A_1,A_2,C_1,C_2,\{\phi_r\}\right) = \int \frac{1}{2}(dA_2+C_2)\wedge *(dA_2+C_2) + A_1\wedge dC_2 - C_1\wedge u i_u C_2 -$$
$$-\sum_r \int d^4x\, \bar{\phi}_r\left(\Box_{e_rA_2+g_rA_1}+m_r^2\right)\phi_r\,, \tag{2.25}$$

where ui_u denotes the projection of the 2-form C_2 along u; in components $(ui_uC_2)_{\rho\sigma} = 2u_{[\rho}u^\beta(C_2)_{\beta\sigma]}$. The covariant derivative on the r-th dyon field is defined by $D_r^\mu = \partial^\mu + i(e_rA_2^\mu + g_rA_1^\mu)$. Furthermore we assume vanishing boundary conditions for C_1 and C_2 at $x^\mu u_\mu = -\infty$.

[2] Actually one can use a generic nowhere lightlike vector field $U^\mu(x)$, see [1] for this more general case.

As shown in [1], the equations of motion (and symmetries of the action) determine the auxiliary fields in terms of A_2 and $\{\phi_r\}$ as follows:

$$\begin{aligned} dA_1 + C_1 &= *(dA_2 + C_2)\,, \\ \left(\Box_{e_r A_2 + g_r A_1} + m_r^2\right)\phi_r &= 0\,, \\ i_u C_1 &= i_u C_2 = 0\,, \\ dC_1 &= J_1\left(\{\phi_r\}, A_1, A_2\right), \\ dC_2 &= J_2\left(\{\phi_r\}, A_1, A_2\right). \end{aligned} \tag{2.26}$$

Here $J_2(\{\phi_r\}, A_1, A_2)$ is defined as the Hodge-dual of $i\sum_r e_r \bar{\phi}_r D_r^\mu \phi_r$ + c.c. The first equation is the standard duality relation $F_1 = *F_2$, which allows to eliminate A_1 and implies the Maxwell equation $d * F_2 = J_1$. The second equation is the covariant Klein-Gordon equation for the matter fields, and the remaining equations determine C_1 and C_2 completely, see below. Therefore, there are no unphysical propagating degrees of freedom.

It remains to prove that gauge-invariant correlation functions are independent of the choice of u^μ. We exhibit the proof for the partition function; for generic correlators see [1]. As in eq. (2.22) we can write:

$$\begin{aligned} \int \prod_r \mathcal{D}\phi_r \exp\left[-i\int d^4x\, \bar{\phi}_r \left(\Box_{e_r A_2 + g_r A_1} + m_r^2\right)\phi_r\right] = \int \prod_r \mathcal{D}\mu(\mathbf{J}_{(r)}) \times \\ \times\ e^{i\int J_1\wedge A_2 - J_2 \wedge A_1}\,, \end{aligned} \tag{2.27}$$

where

$$J_1 = \sum_r e_r \mathbf{J}_{(r)}\,, \qquad J_2 = -\sum_r g_r \mathbf{J}_{(r)}\,, \tag{2.28}$$

and the $\mathbf{J}_{(r)}$ represent a network of closed currents corresponding to the dyon field ϕ_r.

Using (2.27) and integrating out A_1 and C_1 one can write the partition function Z_u, associated to the action (2.25), as

$$\begin{aligned} Z_u = \int \mathcal{D}A_2 \mathcal{D}C_2 \prod_r \mathcal{D}\mu(\mathbf{J}_{(r)})\, e^{i\int \frac{1}{2}(dA_2 + C_2)\wedge *(dA_2 + C_2)} \times \\ \times\ \delta(i_u C_2)\, \delta(dC_2 - J_2)\, e^{i\int J_1 \wedge A_2}\,. \end{aligned} \tag{2.29}$$

Together with the boundary condition along $x^\mu u_\mu \to -\infty$, the two constraints appearing in (2.29) fix C_2 uniquely as

$$C_2(u, J_2) = (u^u \partial_\mu)^{-1} i_u J_2\,. \tag{2.30}$$

In order to satisfy the boundary condition the kernel G associated to the inverse operator $(u^\mu \partial_\mu)^{-1}$ has to be defined as

$$G(x) = \Theta(u_\mu x^\mu)\delta^3(\vec{x}_u^{\perp})\,, \qquad u^\mu \partial_\mu G(x) = \delta^4(x)\,, \tag{2.31}$$

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where $\vec{x}_u^{\perp}$ are the three coordinates orthogonal to $u_\mu x^\mu$. The explicit expression of C_2 is given by

$$C_2(u, J_2) = -\sum_r g_r C_{(r)}\,, \tag{2.32}$$

where

$$C_{(r)}(x) = \frac{1}{2} dx^\mu \wedge dx^\nu \varepsilon_{\mu\nu\rho\sigma} u^\rho \int_0^\infty ds \int_{-\infty}^{+\infty} d\tau_r \frac{dy^\sigma_{(r)}}{d\tau_r}(\tau_r)\delta^{(4)} \times \\ \times \left(x - (y_{(r)}(\tau_r) + us)\right), \tag{2.33}$$

and $y_{(r)}$ parametrizes the network of worldlines supported on the set of curves $\gamma_{(r)}$, corresponding to the current network $\mathbf{J}_{(r)}$. The geometric interpretation of $C_{(r)}$ is rather simple. It is PD of a surface $\Sigma_{(r)}(u)$ whose boundary is $\gamma_{(r)}$ and whose generators are all parallel to u^μ. These two requirements specify $\Sigma_{(r)}(u)$ completely. $C_{(r)}$ is, in particular, an integer form.

The integration over C_2 in (2.29) becomes therefore trivial, and the final integration over A_2 has already been performed for the original Schwinger action and led to the result (2.11). Putting everything together we obtain

$$Z_u = \int \prod_r \mathcal{D}\mu(\mathbf{J}_{(r)}) e^{iS_{\rm eff}(J_1, C_2(u, J_2))}\,. \tag{2.34}$$

For a different vector u', since

$$dC_2(u, J_2) = dC_2(u', J_2) = J_2\,, \tag{2.35}$$

one can write

$$C_2(u', J_2) = C_2(u, J_2) + dH_2(u, u')\,, \tag{2.36}$$

where the 1-form $H_2(u, u')$ is a linear combination of the Poincaré Duals of 3-volumes bounded by the surfaces $\Sigma_{(r)}(u) - \Sigma_{(r)}(u')$, weighted by the magnetic charges g_r, see (2.13). It is then clear that by the same mechanism acting in eq. (2.14), the partition function is independent of the choice of u, provided Dirac's quantization condition (2.15) holds.

The same strategy applies to all correlation functions of local (neutral) gauge-invariant observables.

2.3 Reflection positivity

The analysis developed until now was based on a minkowskian formalism, but it is easy to perform a transition to a euclidean formalism. In particular, the Schwinger action S_S^u has to be replaced with its euclidean counterpart, obtained multiplying by i the second and third terms in (2.25), and using everywhere the euclidean metric.

Starting from the full set of euclidean correlation functions of local gauge-invariant observables, provided a set of properties (the Osterwalder-Schrader (O.S.) axioms [19]) are satisfied, one can reconstruct the Hilbert space of states of the vacuum sector, $\mathcal{H}_0$, containing the vacuum state $|\Omega\rangle$, carrying a unitary representation of the (covering of the) Poincaré group, $\tilde{\mathcal{P}}_+^\uparrow$ leaving $|\Omega\rangle$ invariant, and quantum field operators corresponding to the (classical) euclidean fields. Therefore, the full structure of a Relativistic Quantum Field Theory in its vacuum sector can be reconstructed out of the euclidean correlation functions of local observables, provided O.S. axioms hold.

There is also a version of this reconstruction theorem that applies to lattice regularized theories [20].

The two basic O.S. axioms which allow to set up the entire formalism mentioned above are:

- invariance of the euclidean correlation functions under the euclidean group (lattice translations, in the lattice)

- reflection (or O.S.) positivity. In the models considered here reflection positivity can be defined as follows: let $\mathcal{F}_+$ denote the algebra of gauge invariant functions of the euclidean fields, i.e. euclidean observables, with support in the positive time 4-space; let Θ denote reflection w.r.t. the time zero space followed by complex conjugation. Then reflection positivity means that $\forall \mathcal{F} \in \mathcal{F}_+$

$$\langle \mathcal{F}\Theta\mathcal{F}\rangle \geq 0\,, \tag{2.37}$$

where $\langle\ \rangle$ denotes the euclidean expectation value.

The relation between euclidean observables and quantum observables is then given as follows: let $\mathcal{A}_+ \subset \mathcal{F}_+$ denote the polynomial algebra generated by euclidean local observables supported in the positive time 4-space. To each element $\mathcal{O} \in \mathcal{A}_+$ we associate a vector $|\mathcal{O}\rangle \in \mathcal{H}_0$. The scalar product between such vectors is given by

$$\langle \mathcal{O}|\mathcal{O}'\rangle = \langle \mathcal{O}'\Theta\mathcal{O}\rangle\,. \tag{2.38}$$

Let $\mathcal{O} = \prod_i \mathcal{O}_{x_i^0}$, where $\mathcal{O}_{x_i^0}$ is an euclidean local observable with support at fixed time x_i^0. Then, for $0 \leq x_j^0 < x_{j+1}^0$, we have

$$|\mathcal{O}\rangle = \prod_i \hat{\mathcal{O}}_{x_i^0}|\Omega\rangle\,, \tag{2.39}$$

where $\hat{\mathcal{O}}_{x_i^0}$ is the quantum field operator corresponding to the (classical) euclidean field $\mathcal{O}_{x_i^0}$. If we denote by H the hamiltonian, generator of time translations, then formally $\hat{\mathcal{O}}_{x^0} = e^{-x^0 H}\hat{\mathcal{O}}_0 e^{x^0 H}$, where $\hat{\mathcal{O}}_0$ is a standard "time-zero quantum field operator". [An analogous relation holds between euclidean charged fields and quantum charged fields discussed in the next section: for a more detailed discussion of these methods see [12] and references therein].

Let us show that at a formal level the QFT of dyons defined above satisfies euclidean invariance and reflection positivity. For the euclidean $\mathbf{Z}_4$-theory of dyons invariance under the euclidean group has been established above; in fact, the action S_S^u is manifestly invariant under the euclidean group and the dependence on the fixed vector u^μ has been shown to be spurious at the quantum level.

Reflection positivity can be proved by standard arguments [20]. We use gauge invariance to set the temporal component of the gauge fields to zero. Furthermore we choose u_μ along the time direction, $u_\mu = (1, \vec{0})$. Integration over C_1 sets then $C_2^{0i} = 0$ and the integration over C_2 reduces to an integration over $C_2^{ij} \equiv c^{ij}$. The integration measure becomes then $\mathcal{DM} \equiv \mathcal{D}\vec{A}_1 \mathcal{D}\vec{A}_2 \mathcal{D}c \prod_r \mathcal{D}\phi_r$. Setting

$$\widetilde{S}\left(\vec{A}_1, \vec{A}_2, c, \{\phi_r\}\right) \equiv S_S^u\Big(A_1, A_2, 0, C_2, \{\phi_r\}\Big)\Big|_{A_1^0 = A_2^0 = C_2^{0i} = 0},$$

schematically we have:

$$\begin{aligned}\Big\langle \mathcal{F}(A_1, A_2, C_2)\Theta\mathcal{F}(A_1, A_2, C_2)\Big\rangle &= \int \mathcal{DM}\, e^{-\widetilde{S}}\, \mathcal{F}(\vec{A}_1, \vec{A}_2, c)\, \Theta\mathcal{F}(\vec{A}_1, \vec{A}_2, c) \\ &= \int \mathcal{DM}\, e^{-\widetilde{S}_{x^0>0}}\, \mathcal{F}(\vec{A}_1, \vec{A}_2, c) \times \\ &\qquad \times \Theta\left[e^{-\widetilde{S}_{x^0>0}}\, \mathcal{F}(\vec{A}_1, \vec{A}_2, c)\right] \\ &= \left| \int_{x^0>0} \mathcal{DM}\, e^{-\widetilde{S}_{x^0>0}} \mathcal{F}(\vec{A}_1, \vec{A}_2, c)\right|^2 \geq 0\,.\end{aligned}$$

This computation can be made mathematically precise with a lattice regularization.

2.4 SO(2)-dyons

Let us briefly turn to the quantum field theory of SO(2)-dyons. With the same notation adopted for the $\mathbf{Z}_4$-theory, the (manifestly) SO(2)-invariant action proposed by Zwanziger [8] for the Dirac-Maxwell equations (1.3), is constructed as follows.

First of all, to realize the SO(2)-duality group as a manifest invariance, one introduces a doublet of vector potentials $A \equiv \binom{A_1}{A_2}$. Introducing also a constant 4-vector n^μ with $n^\mu n_\mu = -1$, one constructs a 2×2 matrix-valued operator $Q(n)$ which sends 2-forms into 2-forms

$$Q(n) = \begin{pmatrix} *ni_n & \frac{1}{2} - ni_n \\ -\frac{1}{2} + ni_n & *ni_n \end{pmatrix}. \tag{2.40}$$

The action is then given by

$$S_Z^n(A_1, A_2, J_1, J_2) = \int \frac{1}{2}(dA)^T \wedge Q(n) dA + A_1 \wedge J_2 - A_2 \wedge J_1\,, \tag{2.41}$$

where $(\cdot)^T$ denotes transposition, and one assumes the boundary conditions

$$A_\alpha(n^\mu x_\mu = -\infty) = 0\,, \qquad \alpha = 1, 2. \tag{2.42}$$

The effective action turns out to be

$$S^n_{\text{eff}} = -\frac{1}{2}\int J_\alpha \wedge *\Box^{-1} J_\alpha - \varepsilon^{\alpha\beta} J_\alpha \wedge \Box^{-1}\delta C^n_\beta \,, \tag{2.43}$$

where summation over the SO(2) indices α, β is understood, and the 2-forms C^n (Dirac-strings) are given by

$$C^n_\beta = (n^\mu \partial_\mu)^{-1} i_n J_\beta \,, \tag{2.44}$$

where the inverse operator $(n^\mu\partial_\mu)^{-1}$ is defined as in eq. (2.31).

The difference between the SO(2)- and $\mathbf{Z}_4$-theories is clearly exhibited by the corresponding effective actions, (2.43) and (2.11). Choosing $n^\mu = u^\mu$ one has indeed

$$(S^n_{\text{eff}})_{\mathbf{Z}_4} - (S^n_{\text{eff}})_{\text{SO}(2)} = \frac{1}{2}\int \left(J_1 \wedge \delta\Box^{-1} C_2 + J_2 \wedge \delta\Box^{-1} C_1\right) = \frac{1}{2}\int C_1 \wedge C_2 \,,$$

where the last step, following from Hodge decomposition of the laplacian, is formal because of self-interactions which need a regularization, see below. It is eventually this difference which leads to spin-statistics transmutation in the $\mathbf{Z}_4$-theory, but not in the SO(2)-theory.

C^n describes again the (time-evolution of the) Dirac-string, directed along n, and the consistency requirement is therefore the independence of the exponentiated effective action of the choice of n.[3] Since $dC^n_\alpha = J_\alpha$, for a different unit vector n' we have

$$C^{n'}_\alpha = C^n_\alpha + dH_\alpha(n, n') \,, \tag{2.45}$$

where the 1-forms $H_\alpha(n, n')$ are linear combinations of Poincaré Duals of 3-volumes as in eq. (2.36). A simple computation gives

$$\begin{aligned} S^{n'}_{\text{eff}} - S^n_{\text{eff}} = \frac{1}{2}\int \varepsilon^{\alpha\beta} J_\alpha \wedge \delta\Box^{-1} dH_\beta(n, n') &= \frac{1}{2}\int \varepsilon^{\alpha\beta} J_\alpha \wedge H_\beta(n, n') \\ &= \frac{1}{2}\sum_{r,s}(e_s g_r - g_s e_r)\int J_{(r)} \wedge H_{(s)}(n, n') \,, \end{aligned}$$

and since $\int J_{(r)} \wedge H_{(s)}(n, n') \in \mathbb{Z}$, one obtains as consistency condition the Schwinger-Zwanziger quantization condition

$$\frac{1}{2}(e_r g_s - g_r e_s) \in 2\pi\mathbb{Z} \,. \tag{2.46}$$

From the explicit form of S^n_{eff}, eq. (2.43), it is clear that also at the quantum level the SO(2)-duality group is realized as a manifest symmetry, once independence of n has been established.

[3] In Zwanziger's classical action the vector n induces the projection operator $Q(n)$ in the kinetic terms for the gauge fields and has nothing to do with the Dirac-string. Only at the quantum level, i.e. in S^n_{eff}, it acquires the meaning of the direction of the Dirac-string and, only then, it can be identified with u.

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Since Zwanziger's classical action, eq. (2.41), involves only the currents (and not the strings C), the transition to quantum field theory is straightforward. Zwanziger's action for quantum dyon fields $\{\phi_r\}$ reads:

$$S_Z^n(A_1, A_2, \{\phi_r\}) = \int \frac{1}{2}(dA)^T \wedge Q(n) dA - \\ - \sum_r \int d^4x\, \bar{\phi}_r \left(\Box_{e_r A_1 + g_r A_2} + m_r^2\right) \phi_r \,. \tag{2.47}$$

The proof of independence of n of correlation functions of local gauge-invariant observables can be achieved using representations in terms of closed currents, as in $\mathbf{Z}_4$-theories, provided the Schwinger-Zwanziger quantization condition holds.

Again, one can obtain a euclidean formulation for SO(2)-theories by replacing S_Z^n with a euclidean action, obtained multiplying the off-diagonal terms in $Q(n)$ in (2.47) by i and using the euclidean metric.

At a formal level euclidean invariance of local observables is ensured, once their independence of the choice of n has been established; O.S. positivity can be proven with tools similar to the ones used in $\mathbf{Z}_4$-theories, choosing the vector n^μ with vanishing time component and setting $A_\alpha^0 = 0$, using gauge invariance.

3. Gauge-invariant charged fields

The proof of unobservability of the Dirac-string in correlation functions of local observables depends crucially on their representation in terms of *closed* Feynman paths. Closed paths amount, via PD, to conserved currents and current conservation is, in turn, a consequence of invariance under gauge transformations. Gauge invariance is therefore a natural request in the construction of correlation functions for charged fields. This requirement can be fulfilled by means of an ansatz due to Dirac [11]. To illustrate the ansatz (in its euclidean formulation) and to explain why it has to be modified in a QFT of dyons, we exemplify it in the simple model of a complex scalar field, coupled to a gauge field, discussed previously in eqs. (2.17)–(2.23).

Let $E = dy^\mu E_\mu(y)$ denote a 1-form, with support in the time zero 3-space, satisfying $\partial_\mu E^\mu = \delta^4(y)$. More precisely,

$$E^0(y) = 0\,, \qquad E^i(y) = \delta(y^0) E^i(\vec{y})\,, \qquad \partial_i E^i(\vec{y}) = \delta^3(\vec{y})\,, \qquad i = 1, 2, 3 \tag{3.1}$$

with

$$|\vec{E}(\vec{y})| = O\left(\frac{1}{|\vec{y}|^2}\right), \qquad \text{for } |\vec{y}| \to \infty. \tag{3.2}$$

In particular, a rotation-symmetric choice of $\vec{E}$ corresponds to the (classical) electric field generated by a unitary charge located at the origin, $\vec{E}(\vec{y}) = \vec{y}/4\pi|\vec{y}|^3$. One may, however, consider different choices of $\vec{E}$ corresponding to anisotropic spreadings of the electric flux generated by the charge, which fulfill still the decreasing condition (3.2). In particular, one can concentrate all the flux inside a cone $\mathcal{C}$ with apex at the origin, as discussed in [21] relying on previous ideas developed in [22]. This choice will be relevant in our construction below.

We denote by E^x the 3-form hodge-dual to the above defined 1-form E, translated by x, and define the charged fields

$$\begin{aligned} \phi(E^x) &= \phi(x) e^{ie\int E^x \wedge A}\,, \\ \bar{\phi}(E^y) &= \bar{\phi}(y) e^{-ie\int E^y \wedge A}. \end{aligned} \tag{3.3}$$

Euclidean correlation functions of charged fields are then given by

$$\left\langle \prod_{i=1}^{n} \phi(E^{x_i}) \prod_{j=1}^{n} \bar{\phi}(E^{y_j}) \right\rangle ,$$

and by charge conservation they vanish if the numbers of fields ϕ and $\bar{\phi}$ are different.

At a formal level one can show that the (mixed) euclidean correlation functions of charged fields and neutral observables defined according to the above prescription satisfy a variant of O.S. axioms, in particular translation invariance and reflection positivity. These axioms allow then to reconstruct a Hilbert space of physical states labelled by E, $\mathcal{H}(E)$, carrying a unitary representation of translations, U_E. They allow also to reconstruct quantum charged field operators $\hat{\phi}(E^x)$, $\hat{\bar{\phi}}(E^y)$ (with charges $+e$ and $-e$, respectively) and quantum observables acting on $\mathcal{H}(E)$.

Furthermore, from the vanishing of correlation functions of non-zero total charge, it follows that the Hilbert space $\mathcal{H}(E)$ splits into a direct sum of subspaces $\mathcal{H}_q(E)$ with fixed electric charge qe, $(q \in \mathbb{Z})$:

$$\mathcal{H}(E) = \oplus_q \mathcal{H}_q(E)\,.$$

All these formal considerations can be made mathematically precise in a lattice-regularized version [23].

One may ask if the Hilbert spaces $\mathcal{H}_q(E)$ and $\mathcal{H}_q(E')$ corresponding to different choices of the 1-form E are orthogonal to each other. To analyse this problem one should consider correlation functions of the fields $\phi(E)$, $\phi(E')$ and their complex conjugates. Consider e.g. the two point function $\langle \phi(E^x)\bar{\phi}(E'^y)\rangle$. In a lattice regularization one can show [23] that for large $|x-y|$ it behaves like

$$\left\langle \phi(E^x)\bar{\phi}(E'^y)\right\rangle \sim e^{-c\int (E^x - E'^y)\wedge *\Delta^{-1}(E^x - E'^y)} \cdot \frac{e^{-c'|x-y|}}{|x-y|^{3/2}}\,, \tag{3.4}$$

where c and c' are suitable positive constants and Δ is the 4-dimensional laplacian. Hence, the correlation function vanishes at large distances for electric fields E and E' for which the integral

$$\int (E^x - E'^y) \wedge *\Delta^{-1} (E^x - E'^y) \tag{3.5}$$

diverges. This happens if the behaviour at infinity of $\vec{E}$ is different from that of $\vec{E}'$, e.g. if $\vec{E}$ is supported in a cone $\mathcal{C}$ and $\vec{E}'$ is rotation symmetric or supported in a cone $\mathcal{C}'$ not overlapping with $\mathcal{C}$. A little elaboration shows that if $\int(E-E')*\wedge\Delta^{-1}(E-E')$ diverges, then $\mathcal{H}_q(E)\perp\mathcal{H}_q(E')$. Let us notice, by the way, that, since for $E = E'$ the integral (3.5) is finite, the estimate (3.4) proves that the two-point function of the charged fields is finite. A straightforward generalization of this estimate to n-point correlation functions proves that the reconstructed states $\hat{\phi}(E^x)|\Omega\rangle$ are states in the physical Hilbert space of the ultraviolet regularized theory, with finite energy (otherwise the Green function in (3.4) for $x^0 \neq y^0$ would have vanished).

For what concerns the ultraviolet cutoff we remark that in the theory at hand its removal is not under control at the non-perturbative level. For a discussion of this problem at a perturbative level, although in a somewhat different framework, see [24].

This construction of charged fields, based on Dirac's ansatz, breaks down if electric and magnetic dynamical charges coexist, as in QFT of dyons, due to the consistency requirement of unobservability of the Dirac-string. This failure can be immediately understood by noticing that in the effective action (2.11), corresponding to a QED with electric *and* magnetic currents, in euclidean space-time the last term reads

$$i\int J_1 \wedge \delta\Delta^{-1}C_2\,. \tag{3.6}$$

Using a representation in terms of currents analogous to (2.23), one can see that the electric current J_1, appearing in the correlation functions of charged fields constructed according to Dirac's ansatz, involves also contributions due to the "classical electric fields" E. For example, in the correlator $\langle\phi(E^x)\bar{\phi}(E^y)\rangle$ the electric current 3-form, appearing in the effective action, is given by $J_1 = J + e(E^x - E^y)$, where J is associated, via PD, to an open curve with boundary $\{x, y\}$; therefore the total current is again conserved, $d(J + e(E^x - E^y)) = 0$. But the effective action exhibits now a term $\int[J + e(E^x - E^y)] \wedge \delta\Delta^{-1}C_2$, which is obviously not invariant under a change of Dirac-string, $C_2 \to C_2 + dH_2$, because:

$$\int (e(E^x - E^y) + J) \wedge \delta\Delta^{-1}dH_2 = \int [e(E^x - E^y) + J] \wedge H_2 \notin 2\pi\mathbb{Z}\,. \tag{3.7}$$

In other words, this formulation is inconsistent because the 3-current E is *not an integer current*; its integrals over arbitrary manifolds, contrary to what happens for point-like currents J, do not belong to $\mathbb{Z}$. Therefore, Dirac's quantization condition is not sufficient to make the variation (3.7) an integer multiple of 2π.

A naive arrangement to avoid this inconsistency would be the replacement of E^x by a "Mandelstam string" γ^x, a 3-form current which is PD to a (point-like) curve starting from x and reaching infinity, with support at fixed time x^0. Such a current satisfies still

$$d\gamma^x = \delta_x \equiv d^4y\,\delta^4(y-x)\,,$$

as does E^x, but, being an integer current, (3.7) would become an integer multiple of 2π and the Dirac-string would be unobservable. However, what goes wrong with this recipe is that γ^x violates the condition (3.2), because it is a δ-function on an infinite line. As a consequence incurable infrared divergences would appear. Consider e.g. again the two-point function $\langle \phi(\gamma^x)\bar{\phi}(\gamma^y)\rangle$. From estimates like (3.4) it can be shown that every Mandelstam string γ^x carries an infinite positive self-energy ($\sim \int \gamma^x \wedge *\Delta^{-1}\gamma^x$), and that the interaction energy between two strings with opposite charges ($\sim \int \gamma^x \wedge^* \Delta^{-1}\gamma^y$) is infinite and negative. The reason for these divergences is that the Mandelstam strings have infinite length and that the corresponding electric currents do not decay sufficiently fast at infinity, i.e. they violate (3.2).

The diverging self-energies could be eliminated via multiplicative renormalization, but, since the diverging interaction terms depend on the distance $|x-y|$, their renormalization would spoil O.S. positivity, preventing the reconstruction of charged quantum fields.

A solution to these problems has been proposed in [12]: accordingly one has to replace a fixed Mandelstam-string γ^x by a sum over fluctuating Mandelstam-strings, each one at fixed time x^0, weighted by an appropriate measure $\mathcal{D}\nu(\gamma^x)$. This measure has been constructed in [12] and it is supported on strings which fluctuate so strongly that, with probability 1, the interaction energy between two strings is finite, even for an infinite length.

The two-point function for charged fields should then be defined by

$$\left\langle \int \mathcal{D}\nu(\gamma^x)\, e^{ie\int \gamma^x \wedge A}\phi(x) \int \mathcal{D}\nu(\gamma^y)\, e^{-ie\int \gamma^y \wedge A}\bar{\phi}(y) \right\rangle. \tag{3.8}$$

It has been shown (in euclidean space-time with a lattice cutoff) that there exists a complex measure $D\nu_E(\gamma^x)$ such that

(i) the correlation functions for charged fields constructed in this way satisfy formally the (lattice version of the) O.S. axioms

(ii) at large distances, up to a multiplicative renormalization,

$$\int \mathcal{D}\nu_E(\gamma^x)\, e^{ie\int \gamma^x \wedge A} \sim e^{ie\int E^x \wedge A}, \tag{3.9}$$

where E^x is the "classical" rotation invariant Coulomb field. Hence, on large scales the fluctuating Mandelstam strings produce a phase factor which exhibits the same infrared behaviour as the one appearing in the Dirac ansatz (this has been verified in [12] in gaussian approximation).

By inspection of the explicit construction, one can infer that a straightforward modification of the recipe gives rise to measures $\mathcal{D}\nu_E(\gamma^x)$ which at large distances behave as in (3.9), but where E^x is an electric field supported in a cone $\mathcal{C}$ with corner in $\vec{x}$.[4]

From correlation functions like (3.8) one can reconstruct a quantum field operator $\hat{\phi}(E^x)$ corresponding to the euclidean field

$$\int \mathcal{D}\nu_E(\gamma^x) e^{ie\int \gamma^x \wedge A} \phi(x)\,. \tag{3.10}$$

Point (ii) above implies that the reconstructed states $\hat{\phi}(E^x)|\Omega\rangle$ are states in the physical Hilbert space in the ultraviolet regularized theory with finite energy, as discussed after eq. (3.4).

It is clear how to adapt this construction to dyon quantum fields: we fix an electric field configuration E satisfying (3.1) and (3.2); to this configuration we associate a measure on Mandelstam strings $\mathcal{D}\nu_E(\gamma^x)$ as above. The euclidean correlation functions of the field

$$\int \mathcal{D}\nu_E(\gamma^x)\phi_r(x)\, e^{i\int \gamma^x \wedge \varepsilon^{\alpha\beta} e_{r\alpha} A_\beta} \equiv \phi_r(E^x) \tag{3.11}$$

allow (formally) the reconstruction of a quantum field operator $\hat{\phi}_r(E^x)$, acting on a Hilbert space $\mathcal{H}^r(E)$. It creates dyon states with a dressing cloud of soft "photons", whose infrared behaviour is encoded by E. Here we have set $e_{r\alpha} = (e_r, -g_r)$, so that $\varepsilon^{\alpha\beta} e_{r\alpha} A_\beta = e_r A_2 + g_r A_1$.

This construction holds in the $\mathbf{Z}_4$- as well as in the SO(2)-quantum field theories of dyons.

Vanishing of all correlation functions of non-zero total charge implies that $\mathcal{H}^r(E)$ splits into a direct sum of superselection sectors $\mathcal{H}^r_q(E)$, $q \in \mathbb{Z}$, with electric charge qe_r and magnetic charge qg_r. The field operator $\hat{\phi}_r(E^x)$ maps the vacuum sector, $\mathcal{H}^r_0(E)$, to $\mathcal{H}^r_1(E)$.

More generally, if we consider, in the models discussed here, correlation functions of several species $r = 1, \ldots, N$ of dyons, the total Hilbert space for fixed E is a direct sum of the Hilbert spaces $\mathcal{H}^r(E)$, because the currents associated to each species are individually conserved.

We can now give a functional integral representation for the two-point function of the charged dyon fields defined in (3.11). For the $\mathbf{Z}_4$-theory we use the euclidean version of the Schwinger action, S^u_S (2.25), in the functional integral measure. Apart from an overall normalization we have

$$\begin{aligned}\langle\bar{\phi}_r(E^y)\phi_r(E^x)\rangle = \int \mathcal{D}\nu_E(\gamma^y)\mathcal{D}\nu_E(\gamma^x) \int \prod_s \mathcal{D}\phi_s \prod_\alpha \mathcal{D}A_\alpha \mathcal{D}C_\alpha\, e^{-S^u_S} \times \\ \times\ e^{i\int(\gamma^x-\gamma^y)\wedge\varepsilon^{\alpha\beta} e_{r\alpha} A_\beta}\bar{\phi}_r(y)\phi_r(x)\,.\end{aligned} \tag{3.12}$$

[4] In [12, definition 4.2] one has to choose Neumann b.c. at the boundary of $\mathcal{C}$.

We saw already that the independence of the Dirac-string, i.e. of u, of correlation functions is most easily proved in a path-integral representation; this technique applies also to the two-point function at hand, and we do not repeat here the relevant steps, since they add nothing new. However, as we will see in the next two sections, an analysis of spin and statistics is also most easily carried out in such a representation. Since the spin analysis will be performed on the above two-point function, we give here its path-integral representation explicitly.

To obtain it we follow the steps outlined in section 2. First one integrates over the complex scalars according to (2.23); the role of j_{xy} is here played by the 3-current $\gamma^x - \gamma^y$. Then integration over C_1 and A_1 gives the δ-functions for C_2, as in (2.29). The integration over C_2 can then be performed as after (2.29) and fixes it in terms of the currents. The final integration over A_2 gives rise to the (euclidean) effective action of the $\mathbf{Z}_4$-theory, (2.11). The result is

$$\left\langle \bar{\phi}_r(E^y)\phi_r(E^x)\right\rangle = \int \mathcal{D}\nu_E(\gamma^y)\mathcal{D}\nu_E(\gamma^x) \int \mathcal{D}\mu(J_{(r)}) \prod_s \mathcal{D}\mu(\mathbf{J}_{(s)})\, e^{-S_{\text{eff}}(j)} . \tag{3.13}$$

The current 3-form doublet j_α, $dj_\alpha = 0$, is here given by

$$j_\alpha = \sum_s e_{s\alpha} K_s \, , \tag{3.14}$$

with

$$\begin{aligned} K_s &= \mathbf{J}_{(s)} \, , \qquad \text{for } s \neq r \, , \\ K_r &= \mathbf{J}_{(r)} + \gamma_r \, , \end{aligned} \tag{3.15}$$

where

$$\gamma_r \equiv \gamma^x - \gamma^y + J_{(r)} \, . \tag{3.16}$$

We remember that the $\mathbf{J}_{(s)}$ are closed forms of compact support, corresponding to closed paths in the path-integral measure, which come from the N matter determinants, and that the current $J_{(r)}$ is associated to open paths, with endpoints $\{x, y\}$, which comes from the insertion of the fields $\bar{\phi}_r(y)$ and $\phi_r(x)$. This implies that also γ_r is a closed form, corresponding to a boundaryless (non-compact) path which reaches infinity along γ^x and γ^y. In conclusion, the insertion of the two-point function for the r-th field affects only the r-th current K_r, adding a non-compact closed current.

The Dirac-string 2-forms, $C_\alpha \equiv \sum_s e_{s\alpha} C_s$, are again completely fixed by

$$\begin{aligned} dC_s &= K_s \, , \\ i_u C_s &= 0 \, , \end{aligned} \tag{3.17}$$

for all s. Explicitly, the euclidean version of the effective action (2.11) is

$$S_{\text{eff}}(j) = \int \frac{1}{2} j_\alpha \wedge *\Delta^{-1} j_\alpha + i\, j_1 \wedge \delta\Delta^{-1} C_2 \, , \qquad [\mathbf{Z}_4\text{-theory}] \, . \tag{3.18}$$

Notice, in particular, the appearance of the factor i in the last term which will become crucial below. By the way, from (3.13) one sees immediately that the correlator is Dirac-string independent, because, under a change of Dirac-string, the effective action gets shifted by an integer multiple of $2\pi i$, as shown previously. For this reason we indicated as arguments of the effective action only the currents j_α, and not the strings C_α; strictly speaking it is the exponential $\exp(-S_{\text{eff}}(j))$ which is a functional of only the currents.

For the SO(2)-theory the path-integral representation for charged field correlators can be obtained in the same way as for the $\mathbf{Z}_4$-theory. One has to use Zwanziger's classical action, S_Z^n in (2.47), instead of Schwinger's action in the functional integral (3.12), and the functional integral measure is only over the fields ϕ_r and A^α. Proceeding with the same steps as above one arrives to an expression which is identical to (3.13), apart from the fact that the (euclidean) effective action is now the one of the SO(2)-theory, see (2.43):

$$S_{\text{eff}}(j) = \int \frac{1}{2}\, j_\alpha \wedge *\Delta^{-1} j_\alpha + \frac{i}{2}\,\varepsilon^{\alpha\beta} j_\alpha \wedge \delta\Delta^{-1} C_\beta\,, \qquad [\text{SO(2)-theory}]\,. \tag{3.19}$$

Comparing (3.18) with (3.19) we remarked already that the diagonal (real) contributions are identical and that the difference lies entirely in the imaginary parts: since spin-statistics transmutation is related with phase factors only the imaginary parts can, a priori, give rise to such a phenomenon. We remark also that in the absence of magnetic charges we have $j_2 = 0 = C_2$, the imaginary parts disappear in both effective actions and the correlators reduce to the ones of ordinary scalar electrodynamics, with $S_{\text{eff}}(j) = \frac{1}{2}\int j_1 \wedge *\Delta^{-1} j_1$. Since in this case there is no spin-statistics transmutation it is clear that also for dyons the diagonal parts do not induce such a transmutation. That eventually transmutation occurs only in the $\mathbf{Z}_4$-theory, and not in the SO(2)-theory, is related to the different structure of the imaginary parts of the corresponding effective actions.

Notice also that in generic correlation functions each field $\phi_r(E)$ must be accompanied by a field $\bar{\phi}_r(\tilde{E})$ corresponding to an electric distribution $\tilde{E}$ with the same behaviour at infinity as E, because otherwise the correlation functions vanish, due to infrared divergences, as discussed previously.

3.1 An analysis of the effective action

We devote this subsection to an analysis of the effective actions obtained above, since they will play a crucial role in the derivation of spin and statistics.

Using the above parametrizations of currents and strings we can write the (common) real part of the effective actions as

$$\text{Re}\ S_{\text{eff}} = \frac{1}{2}\sum_{s,t}(e_s e_t + g_s g_t)\int K_s \wedge *\Delta^{-1} K_t\,,$$

while their imaginary parts can be written as

$$\text{Im } S_{\text{eff}} = \begin{cases} -\sum_{s,t} e_s g_t \, \Gamma(K_s, K_t)\,, & \mathbf{Z}_4\text{-theory}\,, \\ -\frac{1}{2}\sum_{s,t} (e_s g_t - e_t g_s)\, \Gamma(K_s, K_t)\,, & \text{SO(2)-theory}\,. \end{cases}$$

We introduced here the real bilinear functional of currents

$$\Gamma(K_s, K_t) \equiv \int K_s \wedge \delta \Delta^{-1} C_t\,. \tag{3.20}$$

Actually, Γ is a functional of the currents only if it is defined mod$\mathbf{Z}$, because, as we saw previously, under a change of the string C_t it changes by an integer. As Γ changes by an integer, the two effective actions, each under its appropriate quantization condition for the charges, change by an irrelevant integer multiple of $2\pi i$. Hence it is sufficient that Γ is a functional of the currents, mod integers.

An explicit representation for it, needed below, can be obtained as follows. We parametrize the closed curve l_s (l_t), associated to K_s (K_t), by $x^\mu(\sigma)$ ($y^\mu(\tau)$). We choose $u^\mu = (1,0,0,0)$ as the direction of the Dirac-string; for the two-form C_t we can then use the explicit expression (2.33). Since in four-dimensional euclidean space-time the inverse of the laplacian, Δ^{-1}, is represented by the Kernel $1/4\pi^2 x^2$, we have

$$(\delta \Delta^{-1} C_t)(x) = \frac{1}{4\pi^2} dx^i \, \varepsilon_{ijk} \oint_{l_t} dy^j \int_0^\infty ds\, \partial^k \, \frac{1}{|\vec{x} - \vec{y}(\tau)|^2 + (x^0 - y^0(\tau) - s)^2}\,.$$

The integral over s is elementary, and one obtains

$$\begin{aligned} \Gamma(K_s, K_t) &= \int K_s \wedge \delta\Delta^{-1} C_t = \oint_{l_s} \delta \Delta^{-1} C_t \\ &= \frac{1}{4\pi^2}\varepsilon_{ijk} \oint_{l_s} dx^i \oint_{l_t} dy^j \, \frac{(x-y)^k}{|\vec{x}-\vec{y}|^3} \left(\frac{\pi}{2} + \text{arctg} \frac{x^0 - y^0}{|\vec{x}-\vec{y}|} + \frac{\frac{x^0-y^0}{|\vec{x}-\vec{y}|}}{1 + \left(\frac{x^0-y^0}{|\vec{x}-\vec{y}|}\right)^2} \right). \end{aligned} \tag{3.21}$$

The main properties of Γ, needed below, can be deduced from this formula. First of all we see that $\Gamma(K_s, K_t)$ is *not* symmetric in the interchange of K_s with K_t; it has a symmetric part, represented by the term $\pi/2$ in the integrand, and an antisymmetric part represented by the arct$\,g$ and the third term in the bracket. In the symmetric part of Γ one recognizes easily the term $\frac{1}{2}\#(\vec{l}_s, \vec{l}_t)$, where $\#(\vec{l}_s, \vec{l}_t)$ indicates the (integer) linking number[5] of the spatial projections of the curves l_s and l_t, a crucial feature for what follows. This means that

$$\Gamma(K_s, K_t) = \frac{1}{2}\#(\vec{l}_s, \vec{l}_t) + \Gamma_{as}(K_s, K_t)\,, \tag{3.22}$$

[5] We recall that the linking number of two curves in three-dimensional space is given by $\#(\vec{l}_s, \vec{l}_t) = \frac{1}{4\pi}\varepsilon_{ijk}\oint_{l_s} dx^i \oint_{l_t} dy^j \, \frac{(x-y)^k}{|\vec{x}-\vec{y}|^3}$.

where Γ_{as} indicates the antisymmetric part.[6] This implies in particular that

$$\Gamma(K_s, K_t) + \Gamma(K_t, K_s) \in \mathbb{Z}\,. \tag{3.23}$$

From (3.21) one sees also that Γ_{as} vanishes if l_s and l_t are at equal times, $x^0(\sigma) = y^0(\tau) = \text{const.}$, or if one curve is compact and the other moves to space-like infinity. In the last case, actually, also $\#(\vec{l}_s, \vec{l}_t)$ vanishes.

Finally we note that the property (3.23) allows to rewrite the imaginary parts of the effective actions, apart from integer multiples of 2π, as

$$\begin{aligned} \operatorname{Im} S_{\text{eff}} &= \\ &= \begin{cases} -\sum_{s>t} (e_s g_t - e_t g_s)\, \Gamma(K_s, K_t) - \sum_s e_s g_s \Gamma(K_s, K_s)\,, & \mathbf{Z}_4\text{-theory}\,, \\ -\sum_{s>t} (e_s g_t - e_t g_s)\, \Gamma(K_s, K_t)\,, & \mathrm{SO}(2)\text{-theory}\,. \end{cases} \end{aligned} \tag{3.24}$$

Here we used the Dirac quantization condition (2.15) for the $\mathbf{Z}_4$-theory and the Schwinger-Zwanziger condition (2.46) for the SO(2)-theory. From these formulae one sees eventually that the unique difference in the effective actions of the two theories is represented by an imaginary part which describes the (diagonal) self-interactions of the s-th dyon with itself, i.e. the second sum in the first line in (3.24). It is precisely this term which will give rise to spin-statistics transmutation for *elementary* dyons.

Notice, however, that in the off-diagonal terms, even if formally identical, the coupling constants $(e_s g_t - e_t g_s)$ belong to $2\pi\mathbb{Z}$ for the $\mathbf{Z}_4$-theory, and to $4\pi\mathbf{Z}$ for the SO(2)-theory.

4. Spin of dyon fields

In this section we outline the derivation of the spin of the r-th dyon species, relying on the construction of dyon quantum field operators sketched in the previous section. The technical details of the derivation, which relies on the properties of the functional Γ displayed above, are relegated to the appendix.

First of all we remark that, except for the rotation symmetric choice of E, the rotation group is not unitarily implementable in $\mathcal{H}(E)$. This is due to the fact for a generic rotation $\mathcal{R}$ the behaviour at infinity of the rotated electric distribution, $E_{\mathcal{R}}$, differs from that of E and, as noticed in the previous section, this implies that the Hilbert spaces $\mathcal{H}(E)$ and $\mathcal{H}(E_{\mathcal{R}})$ are orthogonal to each other.

However, since a rotation of an integer multiple of 2π around an arbitrary axis leaves all local observables invariant, in each sector $\mathcal{H}^r_q(E)$ it must be represented by a phase, by Schur's lemma. Hence, if we denote with $\mathrm{U}(2\pi)$ the unitary operator which represents a 2π rotation, one has

$$\mathrm{U}(2\pi)\, \mathcal{H}^r_1(E) = e^{2\pi i s_r(E)}\, \mathcal{H}^r_1(E)\,, \tag{4.1}$$

[6] The fact that the symmetric part of Γ is semi-integer can also be derived directly from the definition (3.20), decomposing Γ in its symmetric and antisymmetric parts, and then using Hodge decomposition of the laplacian and the fact that $\int C_s \wedge C_t$ is integer.

or

$$\mathrm{U}(2\pi)\,\hat{\phi}_r(E^x)\,U^+(2\pi) = e^{2\pi i s_r(E)}\,\hat{\phi}_r(E^x)\,, \tag{4.2}$$

where $s_r(E)$ is identified with the spin mod 1, also called spin-type of the dyon field $\hat{\phi}_r(E^x)$, see [6].

According to standard arguments one expects $s_r(E)$ to be integer or half-integer.

The correct definition of the action of the operator $\mathrm{U}(2\pi)$ requires some care since the support of the field $\phi_r(E^x)$ extends to infinity; an improper definition could retain erroneously contributions from infinity (see [25] for a discussion of the related problems). In dealing with non-local fields a standard procedure is to introduce a localized version of $\mathrm{U}(2\pi)$, denoted by $U_L(2\pi)$. By definition this operator acts trivially outside a spatial sphere of radius $L+1$, centered at the origin, it induces a 2π-rotation inside a sphere of radius L and it interpolates smoothly in between. The operator $\mathrm{U}(2\pi)$ is then defined as the *weak* limit of $U_L(2\pi)$ as $L \to \infty$.

The simplest correlation function which allows to test the spin of the r-th dyon is the two-point function. In fact, according to the above considerations we can write

$$\lim_{L\to\infty} \langle\Omega|\bar{\hat{\phi}}_r(E^y)\,U_L(2\pi)\,\hat{\phi}_r(E^x)\,U_L^+(2\pi)|\Omega\rangle = e^{2\pi i s_r(E)}\langle\Omega|\bar{\hat{\phi}}_r(E^y)\hat{\phi}_r(E^x)|\Omega\rangle\,. \tag{4.3}$$

Our purpose is to compute $s_r(E)$, using this formula. For the correlator at the r.h.s. with $x^0 > y^0$ we have already a convenient euclidean path-representation, given in (3.13); in the correlator at the l.h.s., for fixed L the field $\hat{\phi}_r(E^x)$ appears rotated according to the above prescription for U_L, while the field $\bar{\hat{\phi}}_r(E^y)$ is unchanged. Taking a look at the representation (3.13) we see that, in the limit $L \to \infty$, the currents $\mathbf{J}_{(s)}$ and $J_{(r)}$ in the integrand are also unchanged, because they are of compact support. The unique ingredients which are of non-compact support are the currents γ^x and γ^y, but, since only $\hat{\phi}_r(E^x)$ gets rotated, it is only the curve γ^x which goes over in a curve γ^x_L. For the correlator at the l.h.s. of (4.3) we can therefore write the following euclidean path-representation ($x^0 > y^0$):

$$N_L(E)\int \mathcal{D}\nu_E(\gamma^y)\,\mathcal{D}\nu_{E_L}(\gamma^x_L)\int \mathcal{D}\mu(J_{(r)})\prod_s \mathcal{D}\mu(\mathbf{J}_{(s)})\,e^{-S_{\mathrm{eff}}(j_L)}\,. \tag{4.4}$$

$N_L(E)$ is a normalization constant, depending on L and E, ensuring that in the limit $L \to \infty$ the action of the 2π rotation reduces to a multiplicative phase factor. The rotated Mandelstam strings γ^x_L are weighted by the measure $\mathcal{D}\nu_{E_L}(\gamma^x_L)$, obtained from $\mathcal{D}\nu_E(\gamma^x)$ through a localized 2π rotation as discussed above.

The effective action depends now on the "rotated" currents j_L which are defined precisely as in (3.14), with the unique difference that the curve γ_r is replaced by

$$\gamma^L_r = \gamma^x_L - \gamma^y + J_{(r)}\,. \tag{4.5}$$

One can also write

$$j_{L\alpha} = j_\alpha + e_{r\alpha} S_L\,,$$

where

$$S_L \equiv \gamma_L^x - \gamma^x \tag{4.6}$$

is a closed curve confined to the region $L \leq |\vec{r}| \leq L+1$, which, as $L \to \infty$, becomes infinitely extended and gets placed at infinity.

In comparing the rotated and unrotated correlation functions, respectively, (4.4) and (3.13), one has to evaluate the behaviour of $S_{\text{eff}}(j_L) - S_{\text{eff}}(j)$ as $L \to \infty$. Since, as remarked above, the real part of this difference can not give rise to a change of spin-type, the spin of the r-th dyon species is given by

$$s_r = \frac{1}{2\pi} \lim_{L\to\infty} \operatorname{Im}\left(S_{\text{eff}}(j_L) - S_{\text{eff}}(j)\right), \tag{4.7}$$

provided this limit is independent of the path-integration variables. This turns out to be true, as shown in the appendix, where we give also a heuristic argument suggesting that one can choose $N_L(E)$ such that $N_L(E)\mathcal{D}\nu_{E_L}(\gamma_L^x)\exp[-\operatorname{Re} S_{\text{eff}}(j_L)]$ approaches $\mathcal{D}\nu_E(\gamma^x)\exp[-\operatorname{Re} S_{\text{eff}}(j)]$ as $L \to \infty$. The crucial ingredients of the computation of the limit (4.7) are the linking numbers appearing in the imaginary parts of the effective action, and an appropriate regularization of the (ultraviolet) divergences showing up in the r-th self-interaction in the first formula of (3.24), which is the unique term which eventually gives a non-vanishing contribution. The final result amounts to the difference of the self-linking numbers of two ribbons (see the appendix) and it reads

$$\lim_{L\to\infty} \operatorname{Im}\left(S_{\text{eff}}(j_L) - S_{\text{eff}}(j)\right) = \begin{cases} (\frac{1}{2} \mod \mathbf{Z})\, e_r g_r\,, & \mathbf{Z}_4\text{-theory}\,, \\ 0\,, & \mathrm{SO}(2)\text{-theory}\,. \end{cases} \tag{4.8}$$

So there is no spin-transmutation in the SO(2)-theory, while in the $\mathbf{Z}_4$-theory the r-th dyon carries spin-(type)

$$s_r = \frac{e_r g_r}{4\pi} = \frac{n_{rr}}{2}\,, \tag{4.9}$$

which is integer or half-integer depending on whether the integer n_{rr} appearing in the Dirac quantization-condition (2.15) is even or odd.

4.1 On the spin addition rule for dyons

We conclude this section with a remark on the spin addition rule.

Let us consider in $\mathbf{Z}_4$ theories the state $|\psi\rangle$ corresponding, as explained in section 2, to the euclidean field $\prod_r \phi_r(E_{(r)}^{x_r})$, with the electric distributions $E_{(r)}^{x_r}$ supported in cones with pairwise disjoint supports. We suppose for simplicity that each species of dyons appears once, i.e. $r = 1, \ldots, N$. If the space-positions $\vec{x}_r$ are lying all in a small region and the time-coordinates practically coincide, then one might consider $|\psi\rangle$ as a state representing a “multidyon composite”. We investigate here the spin of this composite.

The computation of the spin-type of $|\psi\rangle$ can be performed with the techniques sketched above. The simplest correlation function which allows to test its spin involves $|\psi\rangle$ and a product of "compensating" fields carrying charges opposite to those of $|\psi\rangle$ and electric field distributions which ensure that the total flux at space infinity decays faster than Coulombic. The computation of the spin reduces again to the evaluation of a limit like (4.7). A priori it involves the self-linking numbers of the ribbons associated to the deformed curves $\vec{\gamma}_r^L$ defined as above, and the linking numbers among all pairs of these curves. Assume for simplicity that also the deformed electric distributions $E_{(r)L}$ have pairwise disjoint supports. In this geometry, as $L \to \infty$, the linking numbers among different curves vanish and only the self-linking numbers of the ribbons contribute to the spin-type of $|\psi\rangle$ which, according to (4.9), is then given by

$$s_{|\psi\rangle} = \frac{1}{4\pi} \sum_r e_r g_r \mod \mathbf{Z} \tag{4.10}$$

i.e. we obtain the standard spin addition rule for $\mathbf{Z}_4$-dyons. For SO(2)-dyons the spin-type of this composite remains again integer.

On the other hand one might consider in a sense the opposite case of a state $|\tilde{\psi}\rangle$ corresponding to the composite euclidean field

$$\int \mathcal{D}\nu_E(\gamma^x) \prod_r \phi_r(x) e^{i\varepsilon^{\alpha\beta} e_{r\alpha} \int \gamma^x \wedge A_\beta}\,, \tag{4.11}$$

where all fields are located at the same point x and carry a common Mandelstam-string γ^x. The derivation of the spin-type of $|\tilde{\psi}\rangle$ follows the strategy developed above and one has again to evaluate a limit like (4.7). As explained in the appendix each functional Γ in formulae (3.24) contributes with a factor of 1/2 to this limit. This gives for $\mathbf{Z}_4$-dyons, according to (3.24),

$$s_{|\tilde{\psi}\rangle} = -\frac{1}{4\pi}\left[\sum_{s>t}(e_s g_t - e_t g_s) + \sum_s e_s g_s\right] = \frac{1}{4\pi}\left(\sum_s e_s\right)\left(\sum_t g_t\right) \mod \mathbf{Z}\,, \tag{4.12}$$

where we used Dirac's quantization condition. This result coincides with that obtained in quantum-mechanical calculations of the spin-type of dyon composites [2, 3]. In general, however, $s_{|\psi\rangle} \neq s_{|\tilde{\psi}\rangle} \mod \mathbf{Z}$, although the two states have the same total electric and magnetic charges.

The above results indicate that at the QFT-level the spin-type of a multi-dyon state does depend not only on its total charges, but also on the specific asymptotic behaviour of the soft-photon clouds surrounding the dyons.[7]

[7] In the framework of the algebraic approach to relativistic QFT this suggests that the spin-type can depend not only on the charge-class [21], but also on the specific superselection sector within that class.

For SO(2)-dyons (3.24) leads instead to

$$s_{|\tilde{\psi}\rangle} = -\frac{1}{4\pi}\sum_{s>t}(e_s g_t - e_t g_s) \mod \mathbf{Z}\,. \tag{4.13}$$

Here there are no diagonal contributions to spin but, a priori, one has now mixed contributions. However, since the relevant quantization condition is the Schwinger-Zwanziger condition we have $e_s g_t - e_t g_s \in 4\pi\mathbb{Z}$ and (4.13) becomes an integer. Therefore, also for these dyon composites there is no spin-transmutation in the SO(2)-theory.

5. Statistics of dyon fields

In this section we perform the analysis of the statistics of r-th dyon field; more precisely, we discuss the sign appearing in the commutation relation

$$\hat{\phi}_r(E^x)\hat{\phi}_r(E'^y) = \pm\hat{\phi}_r(E'^y)\hat{\phi}_r(E^x)\,, \tag{5.1}$$

which holds provided $x^0 = y^0$ and the support of E^x is disjoint from the support of E'^y.

Such a condition is never satisfied if we consider fields corresponding to the same electric distribution E; for this choice of electric distributions, in charged sectors of gauge theories with infrared QED-like behaviour, in [28] it has been proposed to consider only asymptotic commutation relations.

Here we wish to consider the simpler situation described above, where the supports of E and E' are given by disjoint cones. We derive the sign appearing in (5.1) analysing the monodromy properties of the euclidean correlation functions of dyon fields under their exchange. Four-fields vacuum expectation values of the form

$$\langle\Omega|\bar{\hat{\phi}}_r(E^z)\bar{\hat{\phi}}_r(E'^w)\hat{\phi}_r(E^x)\hat{\phi}_r(E'^y)|\Omega\rangle\,, \tag{5.2}$$

with $x^0 \nearrow y^0$, are the simplest correlation functions allowing to determine the statistics of dyon fields with euclidean methods as follows. For $z^0 < w^0 < x^0$ the v.e.v. (5.2) admits a representation as expectation value of euclidean fields given by

$$\begin{aligned}\Big\langle \int \mathcal{D}\nu_E(\gamma^z)\int \mathcal{D}\nu_{E'}(\gamma^w) e^{-i\varepsilon_{\alpha\beta}\int(e_r^\alpha(\gamma^z+\gamma^w))\wedge A^\beta}\bar{\phi}_r(z)\bar{\phi}_r(w)\cdot \\ \cdot\int \mathcal{D}\nu_E(\gamma^x)\int \mathcal{D}\nu_{E'}(\gamma^y) e^{i\varepsilon_{\alpha\beta}\int(e_r^\alpha(\gamma^x+\gamma^y))\wedge A^\beta}\phi_r(x)\phi_r(y)\Big\rangle\,. \end{aligned}\tag{5.3}$$

This expectation value can in turn be rewritten in terms of path-integrals over currents, which involve two additive terms corresponding to the two admissible contractions of the four scalar fields appearing in (5.3):

$$\begin{aligned}\int \mathcal{D}\nu_E(\gamma^z)\mathcal{D}\nu_E(\gamma^x)\mathcal{D}\nu_{E'}(\gamma^w)\mathcal{D}\nu_{E'}(\gamma^y)\int \mathcal{D}\mu(J_{(r)})\mathcal{D}\mu(J'_{(r)})\prod_s \mathcal{D}\mu(\mathbf{J}_{(s)})e^{-S_{\text{eff}}(j)} + \\ + \{x\leftrightarrow y\}\,. \end{aligned}\tag{5.4}$$

Following the notations of (3.14)–(3.16) we have here the insertion of the non-compact curve

$$
\begin{aligned}
\gamma_r &= \gamma^x - \gamma^z + J_{(r)} + \gamma^y - \gamma^w + J'_{(r)}\,, \\
dJ_{(r)} &= \delta_x - \delta_z\,, \\
dJ'_{(r)} &= \delta_y - \delta_w\,.
\end{aligned}
\tag{5.5}
$$

The second term in (5.4) is obtained from the first one by interchanging x and y. Denoting with $\widetilde{j}$ the total current obtained through this interchange, this means in particular that the exponential in the second term is given by $\exp(-S_{\rm eff}(\widetilde{j}))$. $\widetilde{j}$ differs from j only through the insertion of the non-compact curve, $\widetilde{\gamma}_r$, which is obtained from γ_r with the replacement $x \leftrightarrow y$.

As in the derivation of the spin we must be careful in handling the "behaviour at infinity" and, according to the treatment adopted in [26], we define the exchange of the fields $\phi_r(E^x)$ and $\phi_r(E'^y)$ in the above four-point correlation function as follows. We introduce a deformation of E^x and E'^y acting trivially outside a ball of radius $L+1$, exchanging E^x and E'^y within a ball of radius L and interpolating smoothly for intermediate radii. We further require that the support of the deformed electric distributions, E^y_L and E'^x_L, are still disjoint for sufficiently large L. At the level of Mandelstam strings, this deformation maps the currents j and $\widetilde{j}$ appearing in (5.4) into deformed currents j_L and $\widetilde{j}_L$. In particular, we have

$$
\begin{aligned}
\gamma_{rL} &= \gamma^x_L - \gamma^z + \hat{J}_{(r)} + \gamma^y_L - \gamma^w + \hat{J}'_{(r)}\,, \\
d\hat{J}_{(r)} &= \delta_y - \delta_z\,, \\
d\hat{J}'_{(r)} &= \delta_x - \delta_w\,,
\end{aligned}
$$

and similarly for $\widetilde{\gamma}_{rL}$. As $L \to \infty$ we have

$$
\begin{aligned}
\gamma^x_L &\longrightarrow \gamma^y\,, \\
\gamma^y_L &\longrightarrow \gamma^x\,, \\
\gamma_{rL} &\longrightarrow \widetilde{\gamma}_r\,, \\
\widetilde{\gamma}_{rL} &\longrightarrow \gamma_r\,,
\end{aligned}
$$

and eventually

$$
\begin{aligned}
\widetilde{j}_L &\longrightarrow j\,, \\
j_L &\longrightarrow \widetilde{j}\,.
\end{aligned}
$$

We multiply the correlation functions with the above deformed electric distributions and Mandelstam strings by a normalization constant $N_L(E,E')$, playing a role analogous to $N_L(E)$ in (4.4), and finally we take the limit $L \to \infty$. The constant $N_L(E,E')$ should be chosen in such a way that as a result of the above operations one obtains the original correlation function multiplied by a sign $\pm = e^{i2\pi\theta}$, $\theta = 0, 1/2$.

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From a standard argument of the reconstruction theorem one can infer that this is the sign appearing in (5.1), determining the statistics of dyon fields, and θ is their statistics parameter.

The calculations are then similar to those performed to derive the spin. As $L \to \infty$ in the rotated correlator the first (second) term in (5.4) should go over in the second (first) term, apart from the overall phase $e^{i2\pi\theta}$. The limits relevant to determine the statistics are then again the ones related to the imaginary part of the effective action and the calculations, reported in the appendix, give the results:

$$\begin{aligned} 2\pi\theta &= \lim_{L\to\infty} \mathrm{Im}\left(S_{\mathrm{eff}}(j_L) - S_{\mathrm{eff}}(\widetilde{j})\right) \\ &= \lim_{L\to\infty} \mathrm{Im}\left(S_{\mathrm{eff}}(\widetilde{j}_L) - S_{\mathrm{eff}}(j)\right) = \begin{cases} (1/2 \mod \mathbf{Z})\, e_r g_r\,, & \mathbf{Z}_4\text{-theory}\,, \\ 0\,, & \mathrm{SO}(2)\text{-theory}\,. \end{cases} \end{aligned} \tag{5.6}$$

As shown in the appendix the non-vanishing contribution appearing in $\mathbf{Z}_4$-theories comes from the self-linking number of the ribbons associated to (the three-dimensional spatial parts of) the compact closed currents $j_L - \widetilde{j}$ and $\widetilde{j}_L - j$. These ribbons exhibit an odd number of crossings and therefore their self-linking number is odd [13, 27]. (In the simplest case the ribbon is associated to a curve whose projection on an arbitrary two-dimensional plane has the shape of the symbol "∞".)

From (5.6) one derives that only $\mathbf{Z}_4$-dyon theories with $e_r g_r = 2\pi n_{rr}$ and n_{rr} odd exhibit statistics transmutation. The above results are then consistent with the standard spin-statistics connection, even if spin-statistics transmutation occurs.

Actually, since the relevant calculation is reduced to a computation of self-linking numbers, this connection emerges in a form very similar to the "proof" of the spin-statistics theorem proposed by Wilczek and Zee [29] for anyons in $2+1$ dimensions (See also [30] for a weaker analogy with skyrmions). In fact, these authors associate to every euclidean worldline of an anyon a line of electric flux and an infinitesimally displaced line of vorticity; these two lines define a ribbon, and they analyse the spin of the anyon in terms of the self-linking number of a 2π-twist of the ribbon, and the statistics of the anyon in terms of the self-linking number of a two-ribbon exchange. These self-linking numbers are shown to be identical by a simple geometrical argument [27].

Since in dyon theories we are working in $3+1$ dimensions, the matter appears to be quite different. However, as shown before, choosing the Dirac-strings along the time direction the relevant phase factors arise from a projection of the quantum mechanical trajectories of dyons and of Mandelstam strings in a fixed-time 3-space. After this projection the calculations involved in establishing the spin-statistics connection in $3+1$ dimensions become very similar to those of the (2+1)-dimensional case.

As noticed in section 4.1, however, the situation is somewhat different for what concerns the spin addition rule. This is a consequence of the singular infrared behaviour of $(3+1)$-dimensional dyon systems, whose handling requires a cancellation of fluxes at infinity. This feature has no analogue in (2+1)-dimensional systems of anyons and $(3+1)$-dimensional systems of skyrmions.

Finally let us briefly comment on the dependence of our results on the ultraviolet regulator. In theories where dynamical electric and magnetic charges (or dyons) coexist the usual perturbative framework breaks down because it re-introduces a dependence on the Dirac-string. On the other hand, as we anticipated above, at the non-perturbative level the removal of the ultraviolet cutoff is not under control. However, regarding the stability of spin-type and statistics under ultraviolet renormalization we notice the following: (1) The renormalization of the Dirac quantization condition has been discussed in [31] in adapted perturbative frameworks. The results suggest that the quantization holds also for the renormalized charges, and, moreover that the imaginary part of the effective action is not renormalized. (2) The definition of the self-linking numbers in the imaginary part of our effective actions needs a framing procedure which appears as an ultraviolet regularization. However, since the self-linking number itself is a topological invariant, it does not depend on the scale of the regulator and it remains unchanged as the regulator is sent to zero. Under this limit, actually, the whole imaginary part of the effective action is finite.

A. Computation of the dyon spin

In this appendix we estimate the behaviour of

$$\Delta_L \equiv S_{\text{eff}}(j_L) - S_{\text{eff}}(j)\,,$$

in the limit $L \to \infty$, relevant for spin-transmutation of the r-th dyon, proving formulae (4.8).

We begin with the analysis of the real part, which is the same for SO(2) and $\mathbf{Z}_4$-theories. Using (3.18) (or (3.19)) we get

$$\text{Re } \Delta_L = e_{r\alpha} S_L \wedge *\Delta^{-1} j_\alpha + \frac{1}{2} e_{r\alpha} e_{r\alpha} S_L \wedge *\Delta^{-1} S_L\,.$$

Heuristically one can argue that the second term can be compensated by the constant $N_L(E)$, since it depends only on currents weighted by the measures $\mathcal{D}\nu_{E_L}(\gamma_L^x)$ and $\mathcal{D}\nu_E(\gamma^x)$ and that the first term vanishes in the limit $L \to \infty$, by scale arguments. The reason why one expects this behaviour is that at large scales γ_L^x, γ^x and γ^y are weighted by measures peaked around E_L^x, E^x and E^y, respectively, as follows from eq. (3.9). A mean-field treatment then would give the desired result, since $E^x - E^y$

has a dipole-like decay at space-infinity. That the real part of the effective action can not anyway contribute to spin-statistics transmutation has already been anticipated in text.

We turn now to the imaginary part, starting with the $\mathbf{Z}_4$-theory. In the computation which follows we will make repeated use of the properties of the functional Γ derived in subsection 3.1.

For Im $S_{\text{eff}}(j)$ we use the first formula in (3.24). Then Im $S_{\text{eff}}(j_L)$ is given by the same formula with the unique difference that K_r is replaced with $K_r + S_L$, with S_L given in (4.6). Taking advantage of the bilinearity of the functional Γ one gets (a part from an integer multiple of 2π, which from now on is always understood)

$$-\operatorname{Im}\,\Delta_L = \sum_{s\neq r}(e_r g_s - e_s g_r)\,\Gamma(S_L, K_s) + e_r g_r\Big[\Gamma(K_r + S_L, K_r + S_L) - \\ - \Gamma(K_r, K_r)\Big]. \qquad \text{(A.1)}$$

Since for $s \neq r$ $K_s = \mathbf{J}_{(s)}$ is a compact surface and S_L goes to space-like infinity we have

$$\lim_{L\to\infty} \Gamma(S_L, K_s) = 0\,.$$

On the other hand, for the r-th current we have $K_r = \mathbf{J}_{(r)} + \gamma_r$, where $\mathbf{J}_{(r)}$ is compact while $\gamma_r = \gamma^x - \gamma^y + J_{(r)}$ is not. Therefore, for $L \to \infty$, Im $-\Delta_L$ behaves as

$$e_r g_r\Big[\Gamma(\gamma_r + S_L, \gamma_r + S_L) - \Gamma(\gamma_r, \gamma_r)\Big]. \qquad \text{(A.2)}$$

Now we remained with the currents γ_r and $\gamma_r + S_L = \gamma^x_L - \gamma^y + J_{(r)} = \gamma_{rL}$ which correspond to connected curves; this means that both functionals Γ appearing in (A.2) need a proper regularization due to the 3-space intersection points in the integral (3.21), occurring for $\sigma = \tau$.[8] A standard regularization for such coincident curves is given by a framing procedure, in which the curve appearing say in the first argument of Γ gets displaced by an infinitesimal 3-space vector $\vec{\varepsilon}$, orthogonal to the curve. Accordingly we replace (A.2) by the well defined expression

$$\begin{aligned} e_r g_r\Big[\Gamma(\gamma^\varepsilon_r + S^\varepsilon_L, \gamma_r + S_L) - \Gamma(\gamma^\varepsilon_r, \gamma_r)\Big] &= e_r g_r\Big[\Gamma(\gamma^\varepsilon_r, S_L) + \Gamma(S^\varepsilon_L, \gamma_r) + \Gamma(S^\varepsilon_L, S_L)\Big] \\ &= \frac{1}{2}\, e_r g_r\Big[\#(\vec{\gamma}^\varepsilon_r, \vec{S}_L) + \#(\vec{S}^\varepsilon_L, \vec{\gamma}_r) + \#(\vec{S}^\varepsilon_L, \vec{S}_L)\Big] \\ &= \frac{1}{2}\, e_r g_r\Big(\#(\vec{\gamma}^\varepsilon_{rL}, \vec{\gamma}_{rL}) - \#(\vec{\gamma}^\varepsilon_r, \vec{\gamma}_r)\Big). \end{aligned} \qquad \text{(A.3)}$$

With the subscript ε we indicate the framed currents. In the second line we use bilinearity. To obtain the third line we use the fact that the antisymmetric parts of

[8]This regularization is, actually, needed from the beginning for the functional $\Gamma(K_r, K_r)$, evaluated for two identical currents. However, this regularization does not affect the vanishing of the terms discussed so far.

the functionals Γ cancel: $\Gamma_{as}(S_L^\varepsilon, S_L)$ vanishes because the curves S_L^ε and S_L stay at equal (constant) time x^0; for what concerns $\Gamma_{as}(\gamma_r^\varepsilon, S_L)$ and $\Gamma_{as}(S_L^\varepsilon, \gamma_r)$ we observe that the space-intersection points between γ_r and S_L lie along γ^x, a curve which is at constant time x^0. But since also S_L stays at time x^0, the intersection points do not contribute to the two Γ_{as} in consideration. This means that the regularization can be removed and one has then trivially $\Gamma_{as}(\gamma_r, S_L) + \Gamma_{as}(S_L, \gamma_r) = 0$. In conclusion, in the third line above only the (regularized) linking numbers between the spatial parts of the corresponding currents survive. The fourth line follows from bilinearity of the linking number between two curves in three dimensions.

The pair of curves $\vec{\gamma}_r$ and its framed version $\vec{\gamma}_r^\varepsilon$ define what is called a ribbon, and the integer number $\#(\vec{\gamma}_r^\varepsilon, \vec{\gamma}_r)$ defines then the self-linking number of this ribbon, which is a topological invariant (see e.g. [27] and references therein). So what we are computing in (A.3) is the difference between the self-linking numbers of the ribbons $(\vec{\gamma}_{rL}, \gamma_{rL}^\varepsilon)$ and $(\vec{\gamma}_r, \gamma_r^\varepsilon)$. From the geometry of the curves involved it is clear that, as $L \to \infty$, this difference is *odd*. This leads to

$$\lim_{L\to\infty} \mathrm{Im}\ \Delta_L = \left(\frac{1}{2} \ \mathrm{mod}\ \mathbf{Z}\right) e_r g_r\,, \tag{A.4}$$

as stated in the text.

From this calculation it is also clear that in the SO(2)-theory we have $\lim_{L\to\infty}$ Im $\Delta_L = 0$. This is due to the fact that in the imaginary part of the corresponding effective action, (3.24), the self-interaction terms $\Gamma(K_r, K_r)$, which eventually led to the non-vanishing result in (A.4), are absent.

More generally we conclude that the functional $\Gamma(K_s, K_t)$ can contribute to Im Δ_L as $L \to \infty$ with an additive term of $1/2$, and therefore to spin-type, only if both currents are non compact, corresponding to the insertion of charged fields of the species s and t in the correlator, and if one rotates say one charged field of the s-type and one of the t-type. The case considered above corresponds to $s = t = r$.

B. Computation of the dyon statistics

Here we evaluate the limit

$$\lim_{L\to\infty} \mathrm{Im}\left(S_{\mathrm{eff}}(j_L) - S_{\mathrm{eff}}(\widetilde{j})\right),$$

where the currents are specified in section 5, according to the formulae (3.14)–(3.16). We begin with the $\mathbf{Z}_4$-theory. It is convenient to define the closed compact current

$$S_L = \gamma_{rL} - \widetilde{\gamma}_r\,,$$

which for $L \to \infty$ gets placed at infinity, because then for $\mathrm{Im}(S_{\mathrm{eff}}(j_L) - S_{\mathrm{eff}}(\widetilde{j}))$ we can write an expression which is formally identical to (A.1). With precisely the same steps as above, in particular with the same regularization procedure and using the fact that the curves γ^x, γ^y and S_L stay at the same fixed time $x^0 = y^0$, we arrive at the formula (see the third line in (A.3))

$$-\lim_{L\to\infty} \mathrm{Im}\left(S_{\mathrm{eff}}(j_L) - S_{\mathrm{eff}}(\widetilde{j})\right) = \frac{1}{2}\, e_r g_r \lim_{L\to\infty} \Big[\#(\vec{\gamma}_r^{\varepsilon}, \vec{S}_L) + \#(\vec{S}_L^{\varepsilon}, \vec{\gamma}_r) + \\ + \#(\vec{S}_L^{\varepsilon}, \vec{S}_L)\Big]\,. \tag{B.1}$$

This time γ_r is the union of *two* connected curves, see (5.5), and the geometry is such that for large enough L the sum of liking numbers $\#(\vec{\gamma}_r^{\varepsilon}, \vec{S}_L) + \#(\vec{S}_L^{\varepsilon}, \vec{\gamma}_r)$ becomes *even*. On the other hand, the self-linking number of the curve $\vec{S}_L$, i.e. the linking number of the ribbon $(\vec{S}_L^{\varepsilon}, \vec{S}_L)$, is *odd*. This is due to the fact that the projection of $\vec{S}_L$ on an arbitrary two-plane exhibits an odd number of crossings [27]. In the simplest case this projection corresponds to an "eight". This leads to the result

$$\lim_{L\to\infty} \mathrm{Im}\left(S_{\mathrm{eff}}(j_L) - S_{\mathrm{eff}}(\widetilde{j})\right) = \left(\frac{1}{2} \mod \mathbf{Z}\right) e_r g_r\,,$$

quoted in the text.

In the SO(2)-theory the corresponding limit is zero for the reasons quoted above, i.e. the absence of self-interactions.

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